Some computational results on modules connected with Burnside groups of exponent 4

by

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Abstract

Some computational results on the structure of certain modules connected with Burnside groups of exponent 4 are given. Most results are for the four generator case, as the largest case for which the computations were practical; some results are also given for the three and five generator cases. The chief structural properties examined are composition factors and indecomposable summands, but where a module was sufficiently interesting and small enough for this to be practical a submodule lattice was constructed. In addition, computer code that may be used to reproduce some of the results is given.

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1 Introduction

Let $B = B(d, 4)$ denote the Burnside group of exponent 4 on d generators, that is the freest group on d generators such that the fourth power of every element is the identity. Then B/B^2 is elementary abelian of exponent 2 and may be viewed as a vector space over the two-element field. $GL(d, 2)$ thus acts naturally on B/B^2 and we may obtain a group-ring module Nat corresponding to this natural action.

Further modules connected with $B(d, 4)$ may be constructed by extending the automorphisms to B and its lower exponent-2-central series

$$B = P_0(B) \geq P_1(B) \geq \ldots \geq P_i(B) \geq P_{i+1}(B) \geq \ldots \tag{1}$$

where

$$P_{i+1}(B) = [P_i(B), B]P_i(B)^2 \tag{2}$$

and considering the action on

$$L_i = P_{i-1}(B)/P_i(B). \tag{3}$$

If $d \geq 3$ then $B(d, 4)$ has exponent-2-class $3d - 2$, that is

$$P_{3(d-1)} > P_{3d-2} = \langle 1 \rangle. \tag{4}$$

Note that the extension to B will usually be a pre-image of $GL(d, 2)$, but the action on L_i will only be of $GL(d, 2)$.

The computer algebra system MAGMA [1] was used to examine the structure of the following modules connected with Burnside groups of exponent 4:

- the simple or irreducible modules of $GL(n, 2)$ for $n = 3, 4, 5$;
- the principal indecomposable modules of $GL(3, 2)$ and $GL(4, 2)$;
- the action on L_i for $B(4, 4)$.

The action on L_i was obtained using the ANU p-Quotient algorithm. For a description of this algorithm see [2].

In addition, the structure of certain modules associated with the workings of the p-Quotient algorithm were also examined. In calculating the class $i + 1$ quotient given the class i quotient, the p-Quotient algorithm works in part by adding "tails" to commutators of weight i to give commutators of weight $i + 1$, thereby forming a power-commutator presentation for a group called the covering group. From this we may obtain a module isomorphic to $L_i \otimes Nat$. The power-commutator presentation is then made consistent by eliminating redundant generators, and finally exponent checks take the covering group into class $i + 1$. We therefore have two homomorphisms, where Q_{i+1} is the intermediate quotient from the consistency check:

$$L_i \otimes Nat \overset{\text{map 1}}{\longmapsto} Q_{i+1} \overset{\text{map 2}}{\longmapsto} L_{i+1} \tag{5}$$

$L_i \otimes Nat$ and the kernels of maps 1 and 2 were studied for each class of $B(4, 4)$.

Unless stated otherwise, all calculations were done using MAGMA on a Sparc 10/51.

2 The simple modules of $GL(n, 2)$ for $n = 3, 4, 5$

Beginning with Nat, new simple modules were obtained from known simple modules by forming tensor products with Nat and examining the composition factors of the result. The code used for this purpose may be found in Appendix A.

The calculations took 0.1, 2 and 1772 seconds for $n = 3, 4, 5$ respectively, and the output was as follows:

$GL(3,2)$:

```
GModule Nat of dimension 3 with base ring GF(2),
GModule of dimension 3 with base ring GF(2),
GModule of dimension 1 with base ring GF(2),
GModule of dimension 8 with base ring GF(2)
```

$GL(4,2)$:

```
GModule Nat of dimension 4 with base ring GF(2),
GModule of dimension 6 with base ring GF(2),
GModule of dimension 4 with base ring GF(2),
GModule of dimension 20 with base ring GF(2),
GModule of dimension 1 with base ring GF(2),
GModule of dimension 14 with base ring GF(2),
GModule of dimension 20 with base ring GF(2),
GModule of dimension 64 with base ring GF(2)
```

$GL(5,2)$:

```
GModule Nat of dimension 5 with base ring GF(2),
GModule of dimension 10 with base ring GF(2),
GModule of dimension 10 with base ring GF(2),
GModule of dimension 40 with base ring GF(2),
GModule of dimension 5 with base ring GF(2),
GModule of dimension 40 with base ring GF(2),
GModule of dimension 1 with base ring GF(2),
GModule of dimension 24 with base ring GF(2),
GModule of dimension 74 with base ring GF(2),
GModule of dimension 40 with base ring GF(2),
GModule of dimension 280 with base ring GF(2),
GModule of dimension 40 with base ring GF(2),
GModule of dimension 160 with base ring GF(2),
GModule of dimension 160 with base ring GF(2),
GModule of dimension 280 with base ring GF(2),
GModule of dimension 1024 with base ring GF(2)
```

The dual module to Nat occurs as a simple module in each case; where it is necessary to distinguish between them they will be referred to as Nat and Dual respectively. It will be necessary to distinguish between the two 20-dimensional simple modules of $GL(4,2)$, which are dual to each other; these will be referred to as Simples[4] and Simples[7] for their places in the sequence, as this is the designation each has in MAGMA. Simples[4] may be uniquely specified as the 20-dimensional module appearing as a composition factor of Nat tensored with the 6-dimensional simple module.

The largest simple module in each case is known as the *Steinberg module.*

3 The principal indecomposable modules of $GL(3,2)$ and $GL(4,2)$

To each simple module M of $GL(n,2)$ there corresponds a unique indecomposable module having a single minimal submodule isomorphic to M and having the additional property that it is projective. This module is known as the principal indecomposable module corresponding to M.

The principal indecomposable modules for $GL(3,2)$ were constructed by tensoring each simple module with the Steinberg module and examining the indecomposable summands of the result. The code used for this purpose may be found in Appendix B. This method could not be used for $GL(4,2)$ or $GL(5,2)$ as resource limitations made it impractical to calculate the indecomposable summands of any module of dimension much larger than 1000. In the case of $GL(4,2)$ an alternative method was found; however the principal indecomposable modules of $GL(5,2)$ could not be calculated.

Submodule lattices of the principal indecomposable modules of $GL(3,2)$ appear in figures 1 and 2. Since the Steinberg module is its own principal indecomposable and its lattice is very simple this does not appear.

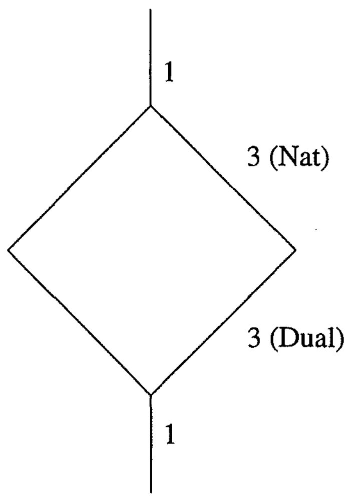

Figure 1: Submodule lattice of the principal indecomposable module corresponding to the trivial module of $GL(3,2)$

If the simple module M appears as a minimal submodule of $K \otimes N$, where N is another simple module, then $\mathrm{PI}(M)$ will appear as a summand of $K \otimes \mathrm{PI}(N)$, where $\mathrm{PI}(H)$ denotes the principal indecomposable module corresponding to H. The Steinberg module was used as a starting point and the principal

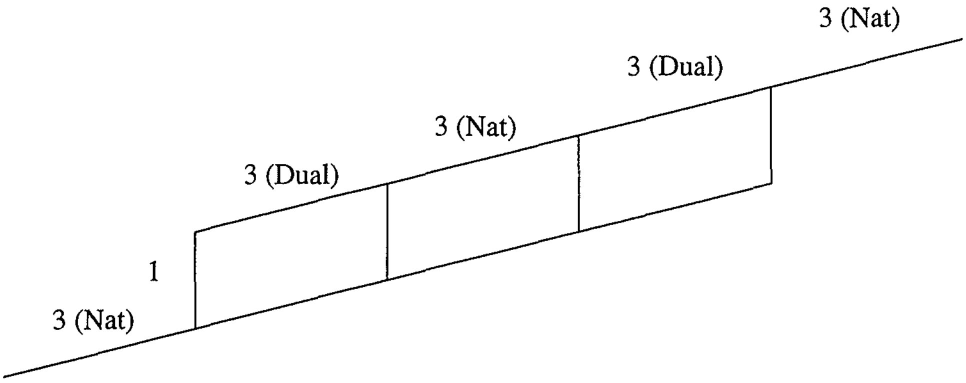

Figure 2: Submodule lattice of the principal indecomposable module corresponding to the natural module of $GL(3,2)$. The principal indecomposable of the Dual is dual to this; its lattice may be obtained by everywhere exchanging Nat and Dual.

indecomposable modules for $GL(4,2)$ were constructed in turn as follows.

1. Nat $\otimes$ 64 (dimension 256) was decomposed yielding PI(Simples[4]) (dimension 192). PI(Simples[7]) is dual to this and so was obtained by taking duals. Time for decomposition: 19 seconds

2. Nat $\otimes$ PI(Simples[4]) (dimension 768) was decomposed yielding the PI of the 14 dimensional simple module (dimension 320). Time for decomposition: 50 minutes.

3. Nat $\otimes$ PI(Simples[7]) (dimension 768) was decomposed yielding the PI of the 6 dimensional simple module (dimension 320). Time for decomposition: 50 minutes.

4. PI(Simples[4]) $\otimes$ 6 (dimension 1152) was decomposed yielding PI(Nat) (dimension 192) and therefore PI(Dual) also upon taking duals. Time for decomposition: 7.8 hours.

5. Finally Nat $\otimes$ PI(Dual) was decomposed to obtain the PI of the trivial module (dimension 448). Time for decomposition: 50 minutes.

Composition factors with multiplicities of the principal indecomposable modules for $GL(4,2)$ appear in table 1.

4 The action on L_i for $B(4,4)$

The action on L_i for $B(4,4)$ was obtained using the ANU p-quotient algorithm, and then studied using MAGMA. The input file used appears in Appendix C.

Simple	Dim. of P.I.	Composition factors with multiplicities of P.I. 1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
1	448	16	4	4	8	8	6	6	
Nat	192	4	5	4	6	4	2	1	
Dual	192	4	4	5	6	4	1	2	
6	320	8	6	6	12	8	2	2	
14	320	8	4	4	8	8	3	3	
Simples[4]	192	6	2	1	2	3	4	2	
Simples[7]	192	6	1	2	2	3	2	4	
64	64								1

Table 1: Composition factors with multiplicities of the principal indecomposable module corresponding to each simple module of $GL(4,2)$

Class	Dim	Composition factors with multiplicities 1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
1	4		1						
2	10		1		1				
3	20						1		
4	55	1			1	2	1		
5	99	3	1	1		2	1	2	
6	84		2		2				1
7	80	2	1			1	2	1	
8	40	2	1		1	2			
9	20							1	
10	10			1	1				

Table 2: Composition factors with multiplicities of the action on L_i

The composition factors and indecomposable summands of each class were calculated, and where the submodule lattice was sufficiently interesting and small enough for this to be practical a lattice was drawn up. Composition factors may be found in table 2 and the lattices in figures 3 to 7.

L_i for $i \in \{1,2,3,4,8,9,10\}$ is indecomposable; L_5 decomposes as $Nat \oplus [95]$, L_6 as $64 \oplus Nat \oplus 6 \oplus Dual(L_{10})$ and L_7 as $Nat \oplus [76]$, where numbers in square brackets denote a module of that dimension and numbers not in square brackets denote the simple module of that dimension.

5 $L_i \otimes Nat$

$L_i \otimes Nat$ was studied in a similar way to L_i for each class. Composition factors may be found in table 3 and indecomposable summands in table 4. Composition factors of some of the indecomposable summands appear in table 5. Some

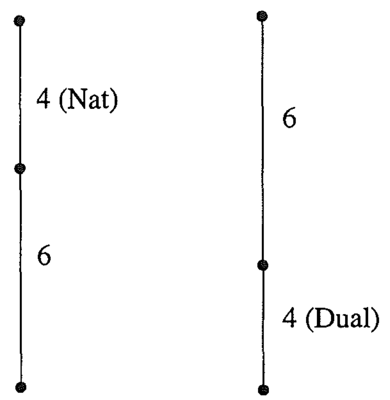

Figure 3: Submodule lattice of L_2 (left) and L_{10} (right), $B(4,4)$.

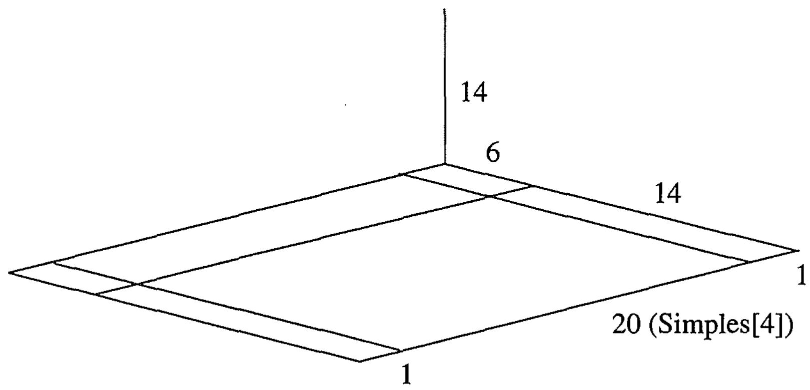

Figure 4: Submodule lattice for class 4, $B(4,4)$.

lattices appear in figures 8 to 11.

6 The kernels

The intermediate quotient Q_i in equation 5 was obtained using the p-Quotient algorithm and then the kernels of the maps 1 and 2 were identified using MAGMA and studied in much the same way as the preceding modules. A MAGMA function written for this purpose may be found in Appendix D while the input file for the ANU p-Quotient algorithm may be found in Appendix E.

Information on the kernels of the first map may be found in table 6 and figure 12, while information on the kernels of the second map is in tables 7–9

Class	Dimension of $Nat \otimes L_i$	Composition factors with multiplicities 1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
1	16		1		2				
2	40		1	1	2		1		
3	80	2		1	2	3	1		
4	220	6	1	2	2	5	2	4	
5	396	8	4	3	8	6	1	4	2
6	336	6	3	4	6	3	6	2	1
7	320	8	2	3	6	8	2	2	1
8	160	4	3	1	2	2	1	4	
9	80			1	2				1
10	40	2		1		1	1		

Table 3: Composition factors with multiplicities of $Nat \otimes L_i$

Class	Dimension of $Nat \otimes L_i$	Indecomposable summands
1	16	indecomposable
2	40	Simples[4] $\oplus$ $[20]_1$
3	80	indecomposable
4	220	[104]$\oplus$ [116]
5	396	(Nat $\otimes$ Nat) $\oplus$ 64 $\oplus$ 64 $\oplus$ [252]
6	336	Dual $\oplus$ (Nat $\otimes$ Nat) $\oplus$ Simples[4] $\oplus$ Simples[4] $\oplus$ $[20]_2$ $\oplus$ 64 $\oplus$ PI(Simples[4])
7	320	(Nat $\otimes$ Dual) $\oplus$ 64 $\oplus$ [240]
8	160	Nat $\oplus$ Nat $\oplus$ Simples[7] $\oplus$ [132]
9	80	(Dual $\otimes$ Dual) $\oplus$ 64
10	40	Dual $\oplus$ [36]

Table 4: Indecomposable summands of $Nat \otimes L_i$. Numbers appearing on their own refer to the simple module of that dimension; numbers in square brackets refer to an indecomposable module of that dimension. Composition factors with multiplicities of these modules may be found in the following table.

Module	Appearing in class...	Composition factors with multiplicities							
		1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
$[20]_1$	2		1	1	2				
$[20]_2$	6		1	1	2				
[36]	10	2				1	1		
[104]	4	2	1	1	2	3	1	1	
[116]	4	4		1		2	1	3	
[132]	8	4	1	1	2	2	1	3	
[240]	7	6	2	3	6	7	2	2	
[252]	5	8	3	3	6	6	1	4	

Table 5: Composition factors with multiplicities of various indecomposable summands of $Nat \otimes L_i$.

Class	Dimension of kernel	Composition factors with multiplicities							
		1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
6	76	2	1	1	3	2		1	

Table 6: Composition factors with multiplicities of the kernel of map 1 for class 6. This module is indecomposable. The remaining kernels are sufficiently small that only a lattice is provided.

and figure 13.

7 The extent to which consistency checks are contained within exponent checks for $B(4,4)$

The usual method of calculating the next class using the p-Quotient algorithm is to add the tails for the next class, carry out consistency checks and eliminate redundant generators, and finally carry out an exponent check and eliminate the remaining redundant generators. The extent to which exponent checks eliminated the need to carry out consistency checks was examined by carrying out the exponent check first. Information on how this was carried out may be found in Appendix F.

For each class, the exponent check brought the number of generators down to the correct number for that class, and consistency checks found no new redundancies. The calculation took about 270 seconds of processor time.

Class	Dimension of kernel	Composition factors with multiplicities 1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
3	10		1		1				
4	5	1		1					
5	86	2	0	1	2	2	1	1	
6	236	6	1	2	3	4	1	3	1
7	230	4	4	2	5	2	3	1	1
8	260	6	1	3	4	5	2	2	1
9	130	4	2	1	1	2	1	3	
10	70				1				1

Table 7: Composition factors with multiplicities of the kernels of the second map

Class	Dimension of kernel	Indecomposable summands
3	10	indecomposable
4	5	1 ⊕ Dual
5	86	$[30] \oplus [56]_1$
6	236	$6 \oplus [56]_2 \oplus 64 \oplus [110]$
7	230	[10] ⊕ Simples[4] ⊕ [20] ⊕ 64 ⊕ [116]
8	260	64 ⊕ [196]
9	130	Nat ⊕ Simples[7] ⊕ [106]
10	70	6 ⊕ 64

Table 8: Indecomposable summands of the kernels of the second map. Numbers appearing on their own refer to the simple module of that dimension; numbers in square brackets refer to an indecomposable module of that dimension. Composition factors with multiplicities of these modules may be found in the following table.

Module	Appearing in class...	Composition factors with multiplicities 1	Nat	Dual	6	14	Simples[4]	Simples[7]	64
[10]	7		1		1				
[20]	7		1	1	2				
[30]	5			1	2	1			
$[56]_1$	5	2				1	1	1	
$[56]_2$	6	2				1	1	1	
[106]	9	4	1	1	1	2	1	2	
[110]	6	4	1	2	2	3		2	
[116]	7	4	2	1	2	2	2	1	
[196]	8	6	1	3	4	5	2	2	

Table 9: Composition factors with multiplicities of various indecomposable summands of kernels of the second map. $[56]_1$ is dual to $[56]_2$.

A Constructing the simple modules

MAGMA code to generate the simple modules given the natural action of $GL(n,2)$ on B/B^2 as a G-Module Nat. $GL(n,2)$ has 2^{n-1} simple modules; this could be introduced as a stopping criterion if greater efficiency were required.

```
Simples:=[Nat];
tobelooped:=[Nat];
while tobelooped ne [] do
   for sim in tobelooped do
       tensorproduct:=TensorProduct(Nat,sim);
       constituents:=Constituents(tensorproduct);
       for con in constituents do
           test:=[IsIsomorphic(con,mods):mods in Simples];
           if true notin test then
               Append(~Simples,con);
               Append(~tobelooped,con);
           end if;
       end for;
       Exclude(~tobelooped,sim);
   end for;
end while;
print Simples;
```

B Constructing the principal indecomposables for $GL(3,2)$

The following code may used to construct the principal indecomposable modules for $GL(3,2)$. The sequence `Simples` generated for $GL(3,2)$ by the code in appendix A is required as input.

```
Indecomposables:=[];
for sim in Simples do
  tensorproduct:=TensorProduct(sim,Simples[4]);
  indecomposables:=IndecomposableSummands(tensorproduct);
  for indeco in indecomposables do
    test:=[IsIsomorphic(indeco,mods):mods in Indecomposables];
    if true notin test then
       Append(~Indecomposables,indeco);
    end if;
  end for;
end for;
print Indecomposables;
```

C Obtaining the action on L_i

ANU p-Quotient input file used to obtain the action on L_i. The same file may be used for each class if edited appropriately.

```
1            #
b44          #
2            #                      { B(4,4), print level 1
10           #<-insert class here   {
1            #                      {
{a,b,c,d}    #
{}           #
4            #

8            #advanced menu

4

18
2            #automorphisms
4

1 1 0 0
0 1 0 0
```

```
0 0 1 0      #matrices
0 0 0 1

0 0 0 1
1 0 0 0
0 1 0 0
0 0 1 0

22      #print matrices
413     #######
413     #######     adjust to upper
422     #######     and lower limits

0
0       #quit
```

D Identifying kernels

Given a module M (argument *super*) and a module N (argument *quotient*) known to be a quotient of M but not created as such in MAGMA,the MAGMA function `IdentifyKernel` returns a submodule H of M such that $N \cong M/H$. First a composition series is calculated for the quotient and then part of the submodule lattice of the super-module is constructed using the factors from the series to restrict the search.

```
IdentifyKernel:=function(super,quotient)

t:=Cputime();
CompFactors:=CompositionFactors(quotient);
print "Composition length of quotient:",#CompFactors;
levels:=[[super]];

for i in [1..#CompFactors] do

    print "starting level", i;
    if levels[i] eq [] then return false; end if;
    levels[i+1]:=[];
    for mods in levels[i] do
        newmods:=MaximalSubmodules(mods);
        for nw in newmods do
            quot:=quo<mods|nw>;
            if IsIsomorphic(CompFactors[#CompFactors+1-i],quot) then
               test:=[nw eq lev:lev in levels[i+1]];
```

```
                if true notin test then
                    Append(~levels[i+1],nw);
                end if;
            end if;
        end for;
    end for;
    levels[i]:=[];
    print "number of modules at level",i,"is:",#levels[i+1];
    print "time so far is",Cputime(t);

end for;

for mods in levels[#CompFactors+1] do
    quot:=quo<super|mods>;
    if IsIsomorphic(quotient,quot) then
        return mods;
    end if;
end for;

return false;

end function;
```

E The intermediate kernel Q_i

The input file for the p-Quotient algorithm used to find the intermediate quotient. The same file may be used for each class if edited appropriately.

```
1          #
b44        #
2          #                          { B(4,4), print level 1
2          ######<-insert class-1 here {
1          #                          {
{a,b,c,d}  #
{}         #
4          #

8          #advanced menu

6          #next class tails computation
7
3          #########<-insert class here
0
```

```
4

8
3            #########<-insert class here
0

11

4

18
2            #automorphisms
4

1 1 0 0
0 1 0 0
0 0 1 0      #matrices
0 0 0 1

0 0 0 1
1 0 0 0
0 1 0 0
0 0 1 0


22
15            ##########adjust lower limit
15            ######
2000          ######

0
0
```

F Carrying out the exponent check before the consistency check

The following p-Quotient input file carries out the exponent check before the consistency check for class 3 of $B(4,4)$. To carry out the calculation for the next class, omit the two zeros at the end and repeat from the point marked "insert a 6 and loop from here".

The exponent check is carried out by inputing the group with no exponent condition, imposing the exponent 4 condition on one generator and then extending it to the remaining generators using the automorphisms.

```
1
b44
2
2
1
{a,b,c,d}
{}
0     # no exponent condition
8

6     #class 3
18
2
4
1 1 0 0
0 1 0 0
0 0 1 0
0 0 0 1

0 0 0 1
1 0 0 0
0 1 0 0
0 0 1 0

5
2
7   #add and compute tails  ######for next class, insert
0                           ######a 6 and loop from here
0


18

1
x4^4;
17
5
1

19  #close the above relation under automorphisms
15
```

```
11

5   #print level
3   #high

8   #consistency check
0
0

5
1
4

0
0
```

References

[1] Wieb Bosma and John Cannon. *Handbook of* MAGMA *functions.* Department of Pure Mathematics, Sydney University, 1994.

[2] George Havas and M.F.Newman. Application of computers to questions like those of Burnside. In *Burnside Groups (Bielefeld, 1977)*, volume 806 of *Lecture Notes in Math.*, pages 211–230. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

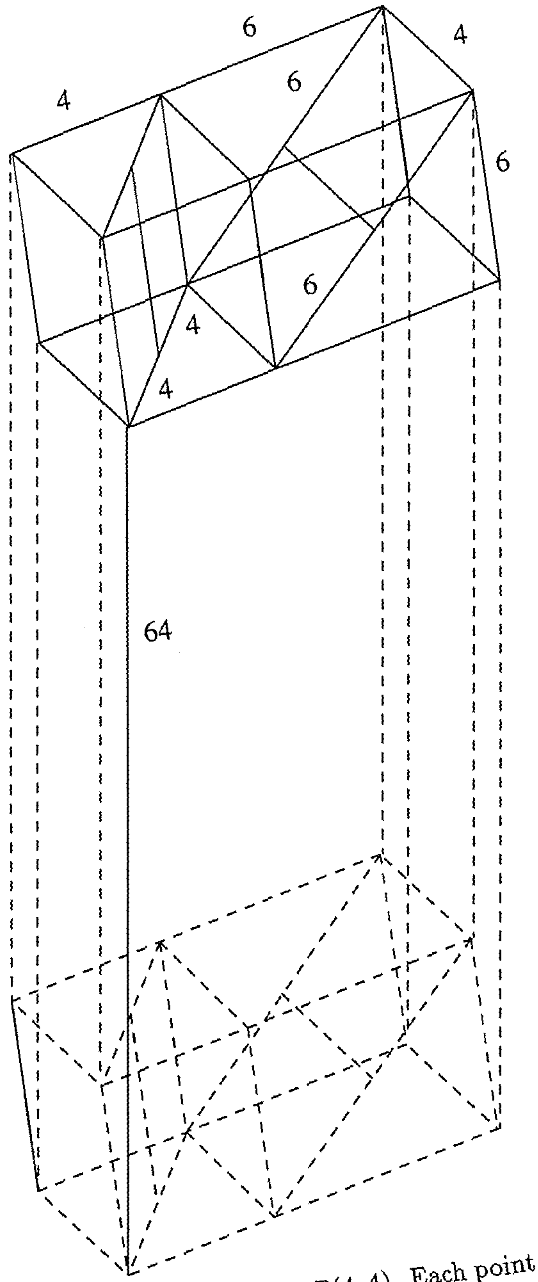

Figure 5: Submodule lattice for L_6, $B(4,4)$. Each point in the upper rectangular prism is connected to the corresponding point in the lower rectangular prism; these lines have been omitted for clarity.

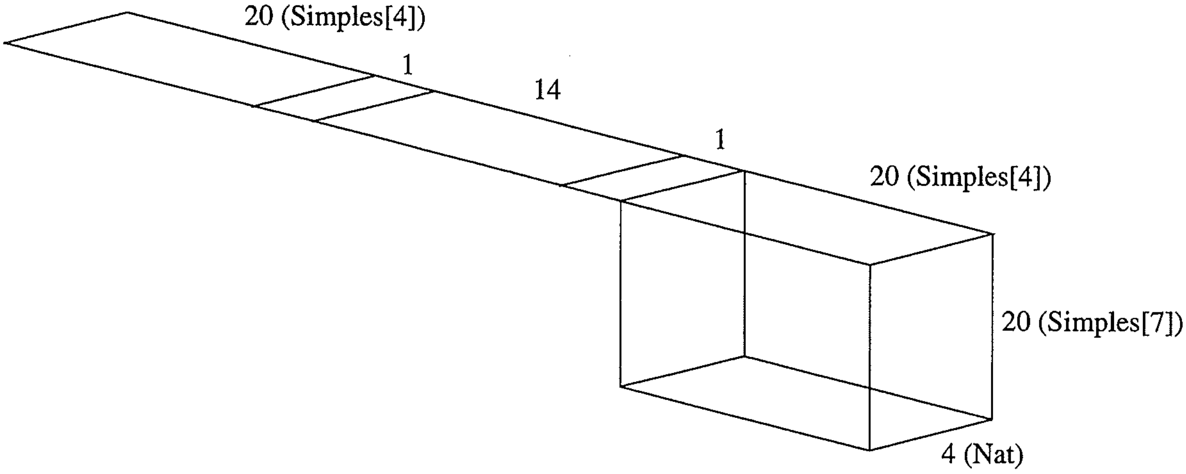

Figure 6: Submodule lattice for L_7, $B(4,4)$.

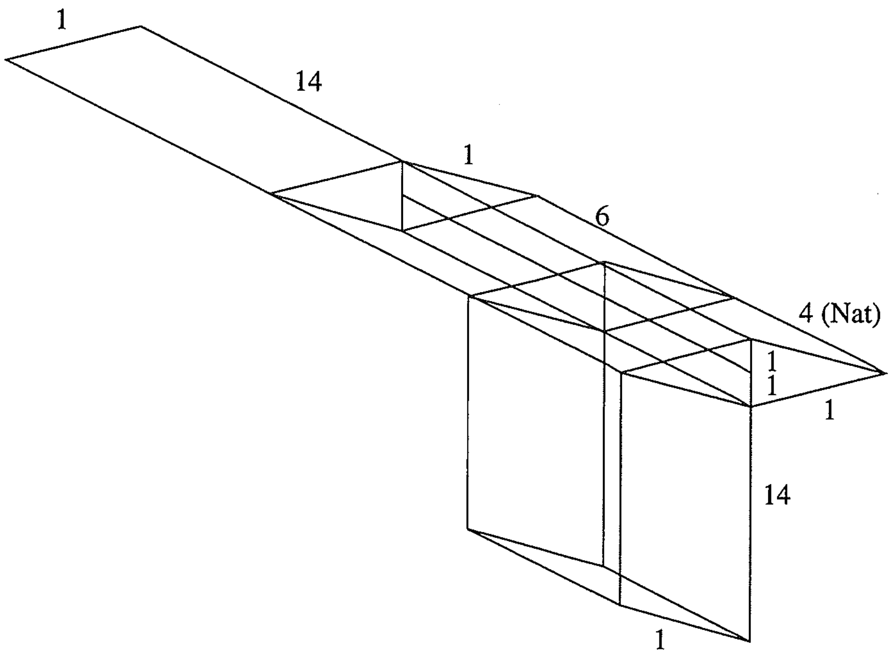

Figure 7: Submodule lattice for L_8, $B(4,4)$.

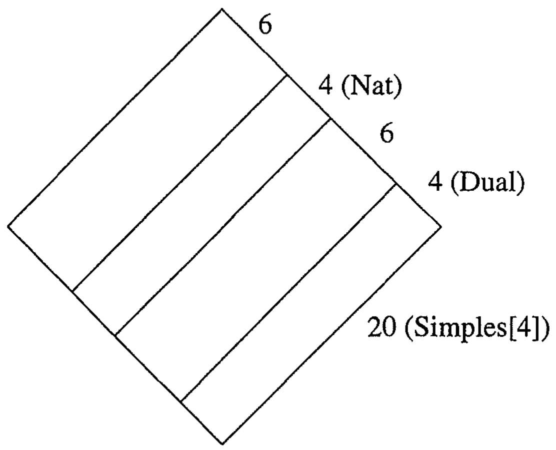

Figure 8: Submodule lattice of $L_2 \otimes Nat$

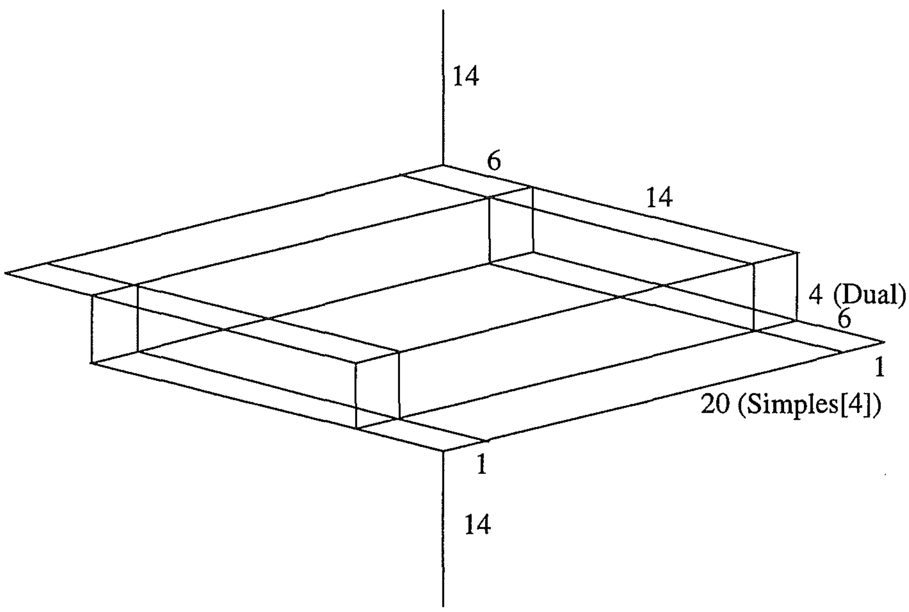

Figure 9: Submodule lattice of $L_3 \otimes Nat$

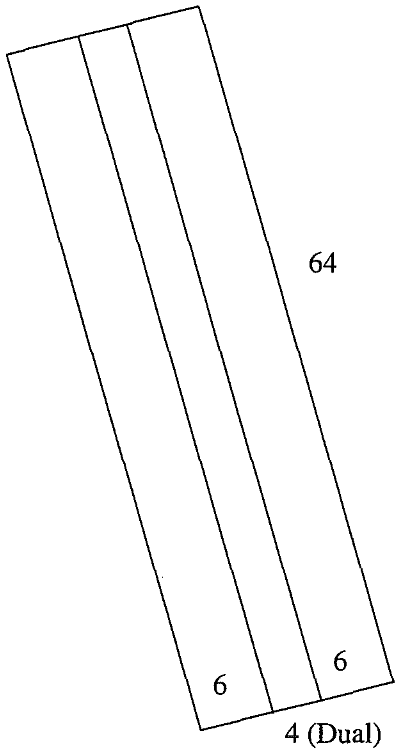

Figure 10: Submodule lattice of $L_9 \otimes Nat$

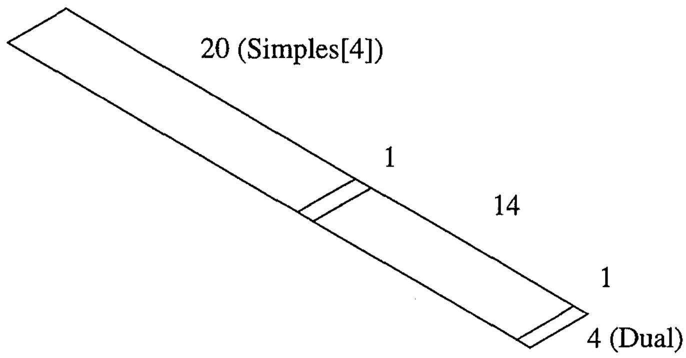

Figure 11: Submodule lattice of $L_{10} \otimes Nat$

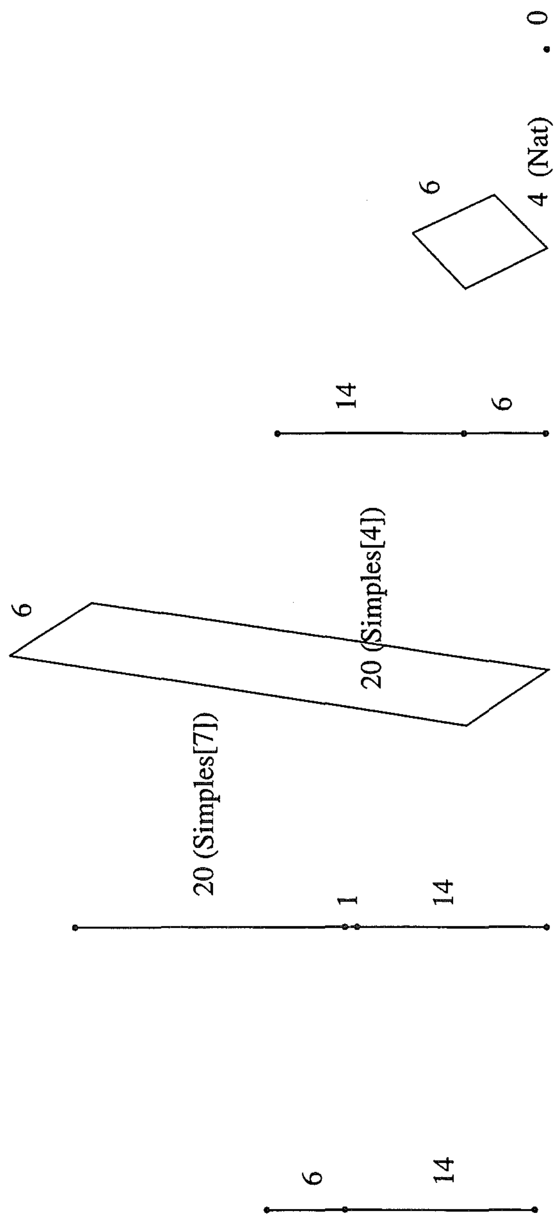

Figure 12: Submodule lattices of kernels of the consistency check map. From left: class 4, class 5, class 7, class 8, class 9 and class 10. Where the lattice is a straight line, the module is indecomposable; where it is a quadrilateral it decomposes as the sum of the two sides.

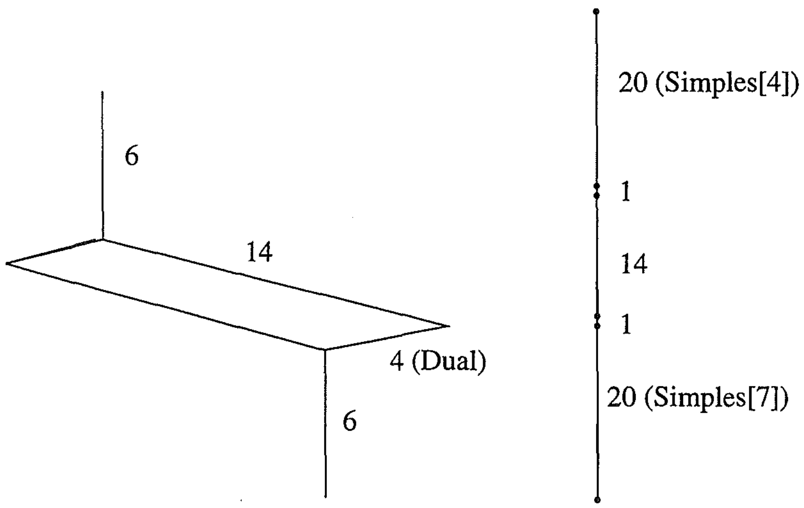

Figure 13: Submodule lattices of the indecomposable summands of the kernel of the second map for class 5.